QCD ON A TRANSVERSE LATTICE

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We present results from a transverse lattice study of low lying mesons. Special emphasis is put on the issue of Lorentz invariant energy-momentum dispersion relations for these mesons. The light-cone wave function for the π obtained in this framework is very close to its asymptotic shape.

1 Introduction

The transverse lattice [1] is an attempt to combine advantages of the light-front (LF) and lattice formulations of QCD. In this approach to QCD the time and one space direction (say x^3) are kept continuous, while the two 'transverse' directions $\mathbf{x}_\perp \equiv (x^1, x^2)$ are discretized. Keeping the time and x^3 directions continuous has the advantage of preserving manifest boost invariance for boosts in the x^3 direction. Furthermore, since $x^\pm = x^0 \pm x^3$ also remain continuous, this formulation still allows a canonical LF Hamiltonian approach. On the other hand, working on a position space lattice in the transverse direction allows one to introduce a gauge invariant cutoff on $\perp$ momenta — in a similar fashion as is done in Euclidean or Hamiltonian lattice gauge theory.

In summary, the LF formulation has the advantage of utilizing degrees of freedom that are very physical since many high-energy scattering observables (such as deep-inelastic scattering cross sections) have very simple and intuitive interpretations as equal LF-time (x^+) correlation functions. Using a gauge invariant (position space-) lattice cutoff in the $\perp$ direction within the Lf framework has the advantage of being able to avoid the notorious $1/k^+$ divergences from the gauge field in LF-gauge which plague many other Hamiltonian LF approaches to QCD [2].

The hybrid treatment (continuous versus discrete) of the long./$\perp$ directions implies an analogous hybrid treatment of the long. versus $\perp$ gauge field: the long. gauge field degrees of freedom are the non-compact A^μ, while the $\perp$ gauge degrees of freedom are compact link-fields. Each of these degees of freedom depend on two continuous ($x^\pm$) and two discrete ($\mathbf{n}_\perp$) space-time variables, i.e. from a formal point of view the canonical $\perp$ lattice formulation is equivalent to a large number of coupled $1+1$ dimensional gauge theories (the long. gauge fields at each $\mathbf{n}_\perp$) coupled to nonlinear σ model degrees of freedom (the link

fields).

2 The color dielectric formulation

For a variety of reasons it is advantageous to work with $\perp$ gauge degrees of freedom that are general matrix fields rather than $U \in SU(N_C)$. First of all, we would like to work at a cutoff scale which is small (in momentum scale) since only then do we have a chance to find low lying hadrons that are simple (i.e. contain only few constituents). If one wants to work on a very coarse lattice, it is useful to imagine introducing smeared or averaged degrees of freedom. Upon averaging over neighboring 'chains' of $SU(N_C)$ fields one obtains degrees of freedom that while they still transform in the same way as the original $SU(N_C)$ degrees of freedom under gauge transformations, they are general matrix degrees of freedom which no longer obey $U^\dagger U = 1$ and $det(U) = 1$. The price that one has to pay for introducing these smeared degrees of freedom are more complicated interactions. The advantage is that low lying hadrons can be described in a Fock expansion (this has been confirmed by calculations of the static quark-antiquark potential [3] and glueball spectra [4]).

Another important advantage of this 'color-dielectric' approach is that it is much easier to construct a Fock expansion of states out of general linear matrix fields than out of fields that are subject to non-linear $SU(N_C)$ constraints.

In the color-dielectric approach the complexity is shifted from the states to the Hamiltonian: In principle, there exists an exact prescription for the transformation from one set of degrees of freedom (here U's) to blocked degrees of freedom $M \equiv \sum_{av} \prod_i U_i$

$$e^{-S_{eff.}(M)} = \int [dU]\, e^{-S_{can.}(U)} \delta \left(M - \sum_{av} \prod_i U_i \right). \qquad (1)$$

The problem with this prescription is that $S_{eff.}$ is not only very difficult to determine directly, but in general also contains arbitrarily complicated interactions.

A much more practical approach towards determining the effective interaction among the link fields nonperturbatively is the use of Lorentz invariance. This strategy has been used in a systematic study of glueball masses in Ref. [4], where more details can be found regarding the effective interaction. One starts by making the most general ansatz for the effective interaction which is invariant under those symmetries of QCD that are not broken by the $\perp$ lattice. This still leaves an infinite number of possible terms and for practical reasons, only terms up to fourth order in the fields and only local (in the

$\perp$ direction) terms have been included in the Ref. [4]. The coefficients of the remaining terms are then fitted to maximize Lorentz covariance for physical observables, such as the $Q\bar{Q}$ potential (rotational invariance!) and covariance of the glueball dispersion relation. It should be emphasized that these are first principle calculations in the sense that the only phenomenological input parameter is the overall mass scale (which can for example be taken to be the lowest glueball mass or the string tension). The only other input that is used is the requirement of Lorentz invariance.

The numerical results from Refs. [4,3] within this approach are very encouraging:

- with only a few parameters, approximate Lorentz invariance could be achieved for relatively large number of glueball dispersion relations simultaneously [4] as well as for the $Q\bar{Q}$ potential
- the glueball spectrum that was obtained numerically on the $\perp$ lattice for $N_C \to \infty$ is consistent with Euclidean Monte Carlo lattice gauge theory calculations performed at finite N_C and extrapolated to $N_C \to \infty$.

For further details on these very interesting results, the reader is referred to Refs. [4,3] and references therein.

3 Fermions on the $\perp$ lattice

Encouraged by the very successful calculations within the pure glue sector, we proceeded to conduct numerical studies that include fermions [5]. In this framework, states that have meson quantum numbers consist either of a q and a $\bar{q}$ on the same transverse site with no link fields required or of a q and a $\bar{q}$ at an arbitrary $\perp$ separation with a chain of link fields in the $\perp$ direction connecting them. Hopping of the quarks in the $\perp$ direction is accompanied by emission or absorption of link field quanta on the link across which the quarks hop. For each transverse site there is a longitudinal gauge field interaction (very similar to the interaction in QCD_{1+1}) which couples to q and $\bar{q}$ on that site as well as to link fields adjacent to that site. In the color dielectric approach the link fields are also subject to the effective interaction discussed above.

Similar to other lattice field theories with fermions, species doubling also occurs for the $\perp$ lattice action. Of course, one main difference to the Euclidean formulation is that the naive $\perp$ lattice action for fermions on the $\perp$ lattice exhibits only $2^2 = 4$ fold species doubling, since only two directions are discretized. Nevertheless, although species doubling is a less extreme phenomenon here, it is a problem that needs to be addressed.

At this point one has several options to proceed. One obvious possibility is to add a Wilson r-term of the form

$$\delta\mathcal{L}_r = ar\bar{\psi}\partial_\perp^2\psi \tag{2}$$

to the $\perp$ lattice action. Obviously, such a term violates chiral symmetry for finite lattice spacing, but this is just a consequence of the well known Nielson-Ninomya theorem, which states that any local and hermitian action for lattice fermions which is chirally symmetric does necessarily exhibit species doubling.

Within the LF framework there seems to be an alternative way to eliminate 'doublers': The crucial observation is that it is possible to write down a fermion (kinetic) mass term within this framework which is chirally invariant (but nonlocal)

$$\delta\mathcal{L}_m = \delta m^2\bar{\psi}\frac{\gamma^+}{i\partial_-}\psi. \tag{3}$$

Since it is possible to write down a chirally invariant mass term, it is also possible to write down a chirally invariant r-term to remove the doublers. [a]In Ref. [5] we investigated the differences between adding a conventional r-term and such a modified chirally invariant r-term to the $\perp$ lattice action. The main problem is that in the canonical LF approach, where half the fermion degrees of freedom are eliminated using a constraint equation, the usual chiral transformations become dynamical operations and therefore the meaning of the usual chiral symmetry becomes obscure. For further details on this issue see Ref. [5].

Both approaches to fermions on the $\perp$ lattice give rise to two kind of hopping terms for the fermions: one that has the Dirac structure of a vector coupling and which flips the helicity (hereafter referred to as spin-flip hopping) and one which has a scalar Dirac structure and does not flip the helicity (hereafter referred to as r-term). The difference between conventional and modified r-term is how coefficients in the LF Hamiltonian are related to coefficients in the Lagrangian. In the spirit of the color-dielectric approach we regard both coefficients in the Hamiltonian as free parameters.

For numerical reasons, we limited the Fock space to states where q and $\bar{q}$ are on the same $\perp$ lattice site (no link field, 2 particle Fock component) and states where q and $\bar{q}$ are separated by one link (3 particle Fock component), i.e the femtoworm approximation [5].

[a]Of course, it is non-local, which is why this does not contradict the Nielson-Ninomya theorem.

4 Fit of parameters

The parameters in the Hamiltonian are:

- longitudinal gauge coupling
- r-term coupling (hopping without spin flip)
- spin-flip coupling (hopping with spin flip)
- kinetic mass (2 particle sector)
- kinetic mass (3 particle sector). The observables that we studied showed little dependence on this parameter, so we kept it fixed at a constituent mass.
- link field mass. Similar to kinetic mass in 3 particle sector. Furthermore, demanding Lorentz invariance in the pure glue sector, one finds a renormalized trajectory. This makes sense since the $\perp$ lattice scale is unphysical. We keep the link field mass fixed at a value which yields relatively small $\perp$ lattice spacing.

The observables that we studied, including Lorentz invariance, show rather little sensitivity [b]to the precise values of the 3-particle sector masses, which we thus keep at a value corresponding to a constituent mass (about half the ρ mass). This leaves us with 4 free parameters.

As input parameter, we use the physical value of the ρ mass, chiral symmetry in the sense that we demand a small π mass and the physical string tension. As explained in the appendix, demanding only Lorentz invariance would drive G^2/m_ρ^2 to infinity and would thus give rise to string tensions that are inconsistent with phenomenology.

Although the dependence of physical parameters on the input parameters is in general rather complex and non-perturbative, one can understand at a qualitative level how the input parameters influence the relevant physical scales: The physical string tension in the longitudinal direction determines G^2 in physical units.

Modulo mass renormalization due to the Yukawa couplings, the quark mass in the 2-particle Fock component and the gauge coupling are strongly constrained by fitting the physical string tension (which determines G^2 and the center of mass of the π-ρ system. This leaves us with only the r-term and the helicity flip hopping term couplings as parameters to vary. As explained in the

[b]As long as we kept the other parameters floating!

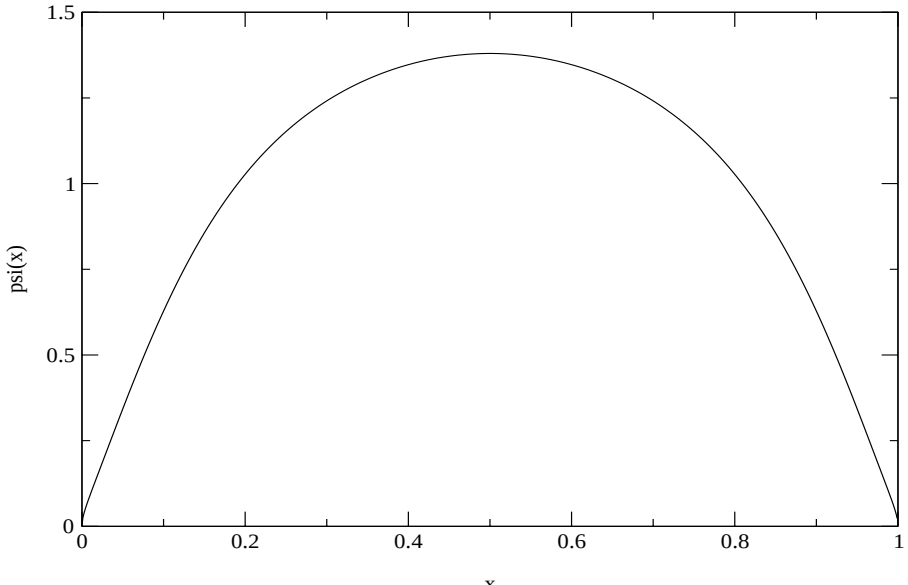

Figure 1: Light-cone wave function for the π obtained on the $\perp$ lattice.

appendix, the helicity flip term is not only responsible for π-ρ splitting within our approximations but also for violations of Lorentz invariance (different $\perp$ lattice spacings in physical units for different mesons).

Using a value of $G^2 \equiv \frac{g^2 N_C}{2\pi} \approx 0.4\ GeV^2$ in physical units, as determined from the string tension in Ref. [4] (which is larger than the one previously used [5,7]), we were able to produce the physical π-ρ splitting with only a relatively small spin-flip coupling. This allows us to chose the r-term large enough so that r-term hopping dominates over spin-flip hopping and therefore violations of Lorentz invariance (as measured by comparing quadratic terms in the dispersion relation of mesons in the π-ρ sector) are only on the order of 20%. The average $\perp$ lattice spacing (from the dependence of the energy on $\mathbf{P}_\perp$) is found to be $a_\perp \approx 0.5 fm$. For the π wave function, we find a shape that is very close to the asymptotic shape ($\phi^\pi_{as.}(x) \propto x(1-x)$). This is surprising if one consider that the lattice spacing is still relatively large and hence the momentum scale is still very low. We should also point out that the shape of our π wave function disagrees with the results from Ref. [6] (the π wave function obtained in Ref. [6] is much more flat than ours). Since the Hamiltonian and Fock space truncation in both works are the same, the only real difference are

the basis functions used to cast the Hamiltonian into matrix form: we used continuous basis functions, while Ref. [6] uses DLCQ. However, it is not clear that this difference alone can explain the different shape of π-wave functions.

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Appendix: perturbative analysis of π-ρ splitting on the $\perp$ lattice.

In order to gain a qualitative understanding about the interplay between different parameters in the $\perp$ lattice Hamiltonian, it is instructive to study a simple model, where one treats the admixture of the 3-partice Fock component to the π and ρ as a perturbation.

To 0^{th} order, i.e. when the coupling between 2 and 3 particle Fock component is turned off, there is no spin dependence of the interactions and the π and ρ are degenerate. Likewise, there is no 'hopping' (i.e. $\perp$ propagation) of mesons and thus energies are independent of $\mathbf{k}_\perp$ giving rise to an infinite $\perp$ lattice spacing (in physical units).

In the next order we treat the coupling between 2 and 3 particle Fock components as a perturbation (note that interactions which are diagonal in the particle number, such as the confining interaction in the longitudinal direction are still treated non-perturbatively). There are two interactions that mix Fock sectors: hopping due to the r-term (without helicity flip) and hopping due to the vector coupling (with helicity flip). In order to understand the effect of these two types of hopping, it is very useful to point out that there is no 'mixing' between these two hopping terms in the femtoworm approximation, i.e. there is a complete cancellation among the various hopping terms where the 2 to 3 transition is caused by say the r-term and the subsequent 3 to 2 transition is caused by the v-term (and the other way round). For $\mathbf{k}_\perp = 0$ this follows trivially from the fact that the $\perp$ lattice Hamiltonian is invariant under rotations around the $z - axis$ by multiples of $\pi/2$, giving rise to conservation of total angular momentum modulo 4. Since $J_z = S_z$ in the 2 particle sector, and S_z can only assume the values -1, 0, and 1, S_z in the 2 particle sector

is conserved. Since 'mixed' hopping would change S_z by one unit, this means that the sum of all contributions from mixed hopping must add up to zero.

For $\mathbf{k}_\perp \neq \mathbf{0}$ the argument is a little more complicated, since rotations also change the direction of $\mathbf{k}_\perp$. However, both the $\perp$ lattice Hamiltonian as well as $\mathbf{k}_\perp$ are invariant under the a sequence consisting of a rotation by π around the z-axis followed by a $\perp$ reflection on the x-axis and then a $\perp$ reflection on the y axis $P_y P_x R_{180}$. As a result, J_z is conserved modulo 2, which still rules out mixing between the r-term and the v-term.

Starting from a basis of 't Hooft eigenstates which are plane waves in the $\perp$ direction and where the $q\bar{q}$ in the 2 particle Fock component carry spins $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$ respectively, one thus finds for the energy in second order perturbation theory

$$\begin{aligned} H = M_0^2 \quad - \quad & M_{1,r}^2 \begin{pmatrix} c_x+c_y & 0 & 0 & 0 \\ 0 & c_x+c_y & 0 & 0 \\ 0 & 0 & c_x+c_y & 0 \\ 0 & 0 & 0 & c_x+c_y \end{pmatrix} \\ + \quad & M_{1,v}^2 \begin{pmatrix} 0 & 0 & 0 & c_y-c_x \\ 0 & 0 & c_x+c_y & 0 \\ 0 & c_x+c_y & 0 & 0 \\ c_y-c_x & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4}$$

Here $M_{1,r}^2$ and $M_{1,v}^2$ are some second order perturbation theory expressions involving matrix elements between 2 and 3 particle states that are eigenstates of the diagonal parts of the Hamiltonian (kinetic + Coulomb), and $c_i \equiv \cos k_i$.

Several general and important features can be read off from this result. First of all, and most importantly, Eq. (4) Shows that the r-term gives rise to a dispersion relation with the same $\perp$ speed of light for the π and the ρ's, while the vector interaction breaks that symmetry. This observation already indicates that it may be desirable to keep the r-term much larger than the spin-flip term. We will elaborate on this point below.

At $k_x = k_y = 0$, the eigenstates of the above Hamiltonian are the $\rho_{\pm 1}$ i.e. $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, with $M^2 = M_{\pm 1}^2 \equiv M_0^2 - M_{1,r}^2$, the $|\rho_0\rangle \equiv |\uparrow\downarrow + \downarrow\uparrow\rangle$, with $M^2 = M_{\pm 1}^2 + M_{1,v}^2$ and the $|\pi\rangle \equiv |\uparrow\downarrow - \downarrow\uparrow\rangle$, with $M^2 = M_{\pm 1}^2 - M_{1,v}^2$ For nonzero $\perp$ momenta, there will in general be mixing among the ρ_{+1} and the ρ_{-1}, but not among the other states since helicity in the 2-particle Fock sector is still conserved modulo 2. Expanding around $\mathbf{k}_\perp = 0$, and denoting

$\bar{M}^2 \equiv M_0^2 - M_{1,r}^2$ one finds to $\mathcal{O}(\mathbf{k}_\perp^2)$ the following eigenstates and eigenvalues

state	$M^2(0)$	$M^2(\mathbf{k}_\perp^2) - M^2(0)$
$\uparrow\downarrow - \downarrow\uparrow$	$\bar{M}^2 - M_{1,v}^2$	$M_{1,r}^2 \frac{k_x^2+k_y^2}{2} + M_{1,v}^2 \frac{k_x^2+k_y^2}{2}$
$\uparrow\downarrow + \downarrow\uparrow$	$\bar{M}^2 + M_{1,v}^2$	$M_{1,r}^2 \frac{k_x^2+k_y^2}{2} - M_{1,v}^2 \frac{k_x^2+k_y^2}{2}$
$\uparrow\uparrow - \downarrow\downarrow$	$\bar{M}^2$	$M_{1,r}^2 \frac{k_x^2+k_y^2}{2} - M_{1,v}^2 \frac{k_x^2-k_y^2}{2}$
$\uparrow\uparrow + \downarrow\downarrow$	$\bar{M}^2$	$M_{1,r}^2 \frac{k_x^2+k_y^2}{2} + M_{1,v}^2 \frac{k_x^2-k_y^2}{2}$

(5)

Eq. (5) illustrates a fundamental dilemma that hampers any attempt to fully restore Lorentz invariance within the femtoworm approximation: $M_{1,v}^2$ not only governs the splitting between the π and the (h=0) ρ but is also responsible for violations of Lorentz invariance among the different helicity states: If one determines the $\perp$ lattice spacing in physical units for each meson separately, by demanding that the $\perp$ speed of light equals 1, one finds for example

$$\begin{aligned} \left.\frac{1}{a_\perp^2}\right|_\pi &= M_{1,r}^2 + M_{1,v}^2 \\ \left.\frac{1}{a_\perp^2}\right|_{\rho_0} &= M_{1,r}^2 - M_{1,v}^2 \end{aligned} \tag{6}$$

i.e. increasing the $\pi - \rho$ splitting is typically accompanied by an increase in Lorentz invariance violation

$$\left.\frac{1}{a_\perp^2}\right|_\pi - \left.\frac{1}{a_\perp^2}\right|_\rho = M_{\rho_0}^2 - M_\pi^2. \tag{7}$$

For the $\rho_{\pm 1}$ the breaking is of a similar scale, plus one also observes an anisotropy in the dispersion relation on the same scale.

Therefore, in order to avoid a large breaking of Lorentz invariance, it will be necessary that

$$M_{r,1}^2 \gg M_{\rho_0}^2 - M_\pi^2. \tag{8}$$

If one keeps the $\pi - \rho$ splitting fixed at its physical value then there are two ways to achieve this condition. One possibility is to simply increase the Yukawa coupling that appears in the r-term. This increase of the r-term tends to decrease the $\perp$ lattice spacing for both π and ρ's and in order to achieve

satisfactory Lorentz invariance (in the sense of uniform $\perp$ lattice spacings) one needs to make the lattice spacing smaller than the Compton wavelength of the ρ meson. However, one cannot make the r-term coupling arbitrarily large because at some point there occurs an instability (tachyonic M^2!). Such instabilities for large coupling are common in the LF formulation of models with Yukawa coupling and might be related to a phase transition (similar to the phase transition in ϕ^4 theory that occurs as the coupling is increased).

Fortunately, there exists another possibility to make these matrix elements large, without increasing the Yukawa couplings. This derives from the fact that the hopping interactions are proportional to $\left(\frac{1}{x} \pm \frac{1}{x'}\right) \frac{1}{x-x'}$, where x (x') are the momenta of the active quark before(after) the hopping. Because of the singularity as $x,\ x' \to 0$, matrix elements of the hopping terms are greatly enhanced if the unperturbed wave functions are large near $x = 0$ and $x = 1$. Since the unperturbed wave functions in the 2 particle Fock component vanish like x^β near $x = 0$, where $\beta \propto \frac{G}{m_q}$, matrix elements of the hopping interaction become very large when one makes $\frac{G^2}{m_q^2}$ very large.

Therefore, the larger one chooses $\frac{G^2}{m^2}$, the more one restores Lorentz invariance of the π and ρ dispersion relations because one can keep the π-ρ splitting fixed while decreasing the coupling of the spin flip interaction. At the same time, keeping the r-term interaction fixed one increases the dominance of the r-term contribution in $\frac{1}{a^2}$ and thus not only reduces the lattice spacing in physical units, but also obtains dispersion relations for the π and the ρ's that look more and more similar — as demanded by Lorentz invariance. Unfortunately, we are not completely free to pick whatever value of $\frac{G^2}{m_q^2}$ we like because m_q^2 and G^2 are largely fixed by the center of mass in the π-ρ system as well as by fitting the physical string tension in the pure glue sector.

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